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# Adaptive Unknown-Input Observers-Based Synchronization of Chaotic Systems for Telecommunication

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**Abstract**—We propose a robust adaptive chaotic synchronization method based on unknown-input observers for master-slave synchronization of chaotic systems, with application to secured communication. The slave system is modelled by an unknown input observer in which, the unknown input is the transmitted information. As in the general observer-based synchronization paradigm, the information is recovered if the master and slave systems robustly synchronize. In the context of unknown-input observers, this is tantamount to estimating the master's states and the unknown inputs. The set-up also considers the presence of perturbations in the chaotic transmitter dynamics and in the output equations (the transmitted signal). That is, the estimator (slave system) must synchronize albeit noisy measurements and reject the effect of perturbations on the transmitter dynamics. We provide necessary and sufficient conditions for synchronization to take place. To highlight our contribution, we also present some simulation results with the purpose of comparing the proposed method to classical adaptive observer-based synchronization (without disturbance rejection). It is shown that additive noise is perfectly canceled and the encoded message is well recovered despite the perturbations.

**Index Terms**—Adaptive control, chaotic communication, nonlinear dynamical systems, observers, state estimation.

## I. INTRODUCTION

ONE of the most popular methods of synchronization is the drive-response (or master-slave) configuration in which there exists a leader (master) which the other system (slave) is required to synchronize with. Since the pioneering work of Pecora and Carroll [1] chaotic synchronization has become a prominent research area in analysis and design of chaotic systems. In part, due to the undisputed utility of the method in secure communication (data encoding and scrambling)—see, e.g., [2], [3]. On the other hand, chaotic oscillators such as the Lorenz, Rössler, Colpitts, Chua, Liu, are easily implementable with common nonlinear electronic components—see, e.g., [4], [5].

In chaos-based communication schemes, the two most applied techniques to encrypt the encoded message are the chaotic

masking method based on masking the transmitted information by adding it to a chaotic signal from the master system [6], and the chaotic modulation method based on modulating one of the drive system parameters by the encoded information [7].

As it has been keenly explained in the tutorial paper [8] master-slave synchronization may be recasted from a control-view point, in the paradigm of observer design or more generally, state and input estimators. Early work on observer-based synchronization includes [9]; more recent work, including parametric uncertainty and adaptation are the interesting papers [10], [11] and the case of systems with unknown disturbances has been studied for instance in [12]. Generally speaking, the slave system (receiver) is considered as an observer of the master system (transmitter). The message is recovered if the systems synchronize their motions. In the case of simple Luenberger-type observers, arguably, the scheme works provided that the systems synchronize. However, the latter is possible if the signal/noise ratio of the carrier relative to the input is considerably large. In other words, the power of the information signal must be several times smaller than that of the chaotic carrier—see [13]. Nevertheless, not only this is a restrictive assumption but in general, such an ad hoc method works poorly as simulations and analytical results show—see [14].

Of particular interest for the use of synchronization of chaotic systems in communication is that of parametric uncertainty. Two problems are to be identified separately: 1) synchronization in spite of parametric uncertainty; 2) asymptotic parameter estimation. It has been mentioned that the literature lacks from a strict analysis on parametric convergence—see [15]–[17].

The method presented in [18] achieves master-slave synchronization under parameter uncertainty at the expense of synchronization mismatch then, an adaptive algorithm is activated to estimate the parameters. The method works *locally*. The article [19] motivated many other works on adaptive synchronization and identification of chaotic systems. For instance it is stressed in [16], [20] that the method of proof in [19] is inadequate: in [16] an alternative proof for parametric convergence is given which relies on La Salle's invariance principle however, the latter may not be used for non-autonomous systems as is the case here. In [20] a proof of convergence of synchronization errors is established following “signal-chasing” arguments standard in adaptive control theory but parametric convergence is not established, it is only observed (for the particular case of the Lorenz system) that parameters converge when the system is in a chaotic or in a periodic regime—this is stressed as an “interesting phenomenon which remains to be further investigated.”

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Another article along similar lines is [15] where a standard adaptive control law is applied and conditions in terms of linear independence of certain vector-fields are presented for parametric convergence. However, such sufficient conditions are not met in general by chaotic systems.

The “interesting phenomenon” observed in [20] has been studied and explained recently in [21] in terms of so-called *persistence of excitation* (PE) which is a necessary condition for parametric convergence under *appropriate* structural conditions, not in general. In Section III-B we explain in certain detail the stabilization mechanism.

In the context of chaotic synchronization *unknown-input* observers is used as follows. As in the general observer-based synchronization paradigm, the slave system is considered as an observer. The unknown input to be reconstructed is the information signal that drives the master dynamics. Besides, in this paper it is assumed that non-vanishing disturbances affect the master’s equations; such perturbations may be generated by un-modelled dynamics. In addition, we assume parametric uncertainty and show convergence of parametric estimates to the true values under appropriate persistency-of-excitation conditions—see [21]. Finally, the scenario in this paper also considers that the measured output from the master system is noisy. To the best of our knowledge chaotic synchronization for communication in such realistic and complete scenario has not yet been solved. For instance, [21], [22] presents adaptive observers which achieve practical estimation (synchronization with small steady-state errors which may be diminished by design); the unknown-input observers in [23] are designed under the assumption of noise-free outputs.

We also present a series of simulation results to compare our approach against classical adaptive-observer-based synchronization (without input estimation and perturbation rejection)—cf. comparison with the classic methodology based on adaptive observer adopted recently in several works [7], [12].

The rest of the paper is organized as follows. In the following section we describe the context and formulate the synchronization problem that we solve; in Section III we present our unknown-input observers, we give stability and convergence proofs and discuss their implementation; in Section IV we present a comparative study in simulations through two concrete examples. We conclude with some remarks in Section V.

*Notation:* The following notation is used throughout this paper.  $|\cdot|$  denotes the absolute value for scalars, the Euclidean norm for vectors, and the induced norm for matrices. We use  $M^+$  for the generalized inverse of a matrix  $M$ . The smallest and largest eigenvalues of  $M$  are denoted by  $\lambda_{\min}(M)$  and  $\lambda_{\max}(M)$  respectively. In particular, we may define  $|M| := [\lambda_{\max}(M^T M)]^{1/2}$ .

## II. CONTEXT AND PROBLEM STATEMENT

Consider nonlinear systems

$$\dot{x} = Ax + Bf_0(x) + Bg_0(x)m(t) + Fd(t) \quad (1a)$$

$$y = Cx + Gd(t) \quad (1b)$$

where  $x \in \mathbb{R}^n$  is the state vector,  $y \in \mathbb{R}^p$  is a measurable output,  $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}^s$  and  $g_0(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{s \times q}$  are once continuously differentiable;  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times s}$ ,  $C \in \mathbb{R}^{p \times n}$ ,  $F \in \mathbb{R}^{n \times r}$  and  $G \in \mathbb{R}^{p \times r}$  are known constant matrices.

The functions  $m : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^q$  and  $d : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^r$  are assumed to be (component-wise) piece-wise constant and (Lebesgue) measurable respectively. It is assumed that there exists  $K_m > 0$  such that

$$\sup_{t \geq 0} |m(t)| \leq K_m. \quad (2)$$

The function  $m$  represents a vector of unknown parameters to be estimated; in the context of synchronization for communication,  $m$  represents the valuable information that is encoded by the dynamics (1a). Respectively, the transmitted signal is  $y$ , which is affected by disturbances and noise, denoted by  $Gd(t)$ . These are to be rejected.

The model (1) covers virtually any chaotic system. Indeed, note that the Duffing, van der Pol, Lorenz, Lü of 3rd and 4th order, Lorenz, Rössler, Chua, ... are of the form

$$\dot{x} = Ax + Bf_0(x). \quad (3)$$

Let us recall some of the cited systems: the **Rössler oscillator**:

$$\dot{x}_1 = -(x_2 + x_3) \quad (4a)$$

$$\dot{x}_2 = x_1 + ax_2 \quad (4b)$$

$$\dot{x}_3 = b + x_3(x_1 - c) \quad (4c)$$

is of the form (3) with

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$f_0(x) = b + x_1x_3.$$

The **Lorenz oscillator**:

$$\dot{x}_1 = \sigma(x_2 - x_1) \quad (5a)$$

$$\dot{x}_2 = rx_1 - x_2 - x_1x_3 \quad (5b)$$

$$\dot{x}_3 = x_1x_2 - bx_3 \quad (5c)$$

is of the form (3) with

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$f_0(x) = \begin{bmatrix} -x_1x_3 \\ x_1x_2 \end{bmatrix}$$

and so are the **Lü oscillator of 4th order**:

$$\dot{x}_1 = a(x_2 - x_1)$$

$$\dot{x}_2 = bx_1 - kx_1x_3 + x_4$$

$$\dot{x}_3 = -cx_3 + hx_1^2$$

$$\dot{x}_4 = -dx_1$$

and the **Chua double scroll system**;

$$\begin{aligned}\dot{x}_1 &= \alpha(x_2 - x_1 - \phi(x)) \\ \dot{x}_2 &= x_1 - x_2 + x_3 \\ \dot{x}_3 &= -\beta x_2\end{aligned}$$

with

$$\phi(x) = \begin{cases} bx + a - b & \text{if } x > 1 \\ ax & \text{if } |x| \leq 1 \\ bx - a + b & \text{if } x < -1 \end{cases}$$

to describe only a few.

**Problem statement.** The synchronization problem consists in designing a slave dynamical system

$$\dot{z} = \Phi(t, y, z, \hat{m}) \quad (6a)$$

$$\hat{x} = h(y, z) \quad (6b)$$

$$\dot{\hat{m}} = \Psi(t, y, z) \quad (6c)$$

such that for any  $r > 0$

$$\lim_{t \rightarrow \infty} |x(t) - \hat{x}(t)| = 0 \quad \forall x(0), \hat{x}(0) \in B_r \times \mathbb{R}^n \quad (7)$$

where  $B_r := \{x \in \mathbb{R}^n : |x| < r\}$ . In particular, it is required to reject the perturbation  $d$  which affects both the system dynamics and the measured output.

To solve the problem stated we rely on the following.

**Boundedness property.** The solutions  $x(\cdot)$  to (1) are forward complete and uniformly bounded.

Boundedness is a common assumption in the literature of observer design yet, it holds for many physical systems such as oscillators.<sup>1</sup> More particularly, chaotic oscillators. The boundedness property allows us to perform the following transformation, some times called “Lipschitz extension”. Let  $\omega_i > 0$  be arbitrarily given for each  $i \in \{1, \dots, n\}$ . Define the compact  $\Omega \subset \mathbb{R}^n$

$$\Omega := \{x \in \mathbb{R}^n : |x_i| \leq \omega_i\}$$

and define the saturation function  $\sigma : \mathbb{R}^n \rightarrow \Omega$  where each component  $\sigma_i$  of  $\sigma$  is given by

$$i \in \{1, \dots, n\} \Rightarrow \sigma_i(x) := \begin{cases} x_i & \text{if } |x_i| \leq \omega_i \\ \text{sgn}(\omega_i)|\omega_i| & \text{otherwise} \end{cases}$$

Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}^s$  and  $g : \mathbb{R}^n \rightarrow \mathbb{R}^{s \times q}$  for each  $x \in \mathbb{R}^n$  as  $f(x) := f_0 \circ \sigma(x)$  and  $g(x) := g_0 \circ \sigma(x)$ . The following observations are in order (the proof of the second is provided in the Appendix):

*Fact 1:*

- 1) for all  $x \in \Omega$  we have  $f(x) = f_0(x)$  and  $g(x) = g_0(x)$ ;
- 2) For each  $\Omega$  there exist positive reals  $K_f$  and  $K_g$  such that for any  $a, b \in \mathbb{R}^n$  and  $w \in \mathbb{R}^s$

$$\begin{aligned}|w^\top [f(a) - f(b)]| &\leq K_f |a - b| |w| \\ |w^\top [g(a) - g(b)]| &\leq K_g |a - b| |w|.\end{aligned}$$

<sup>1</sup>The reader shall not understand systems with periodic solutions. See [24] for several broad definitions of oscillators.

Then, for all  $x \in \Omega$  and  $t \in \mathbb{R}_{\geq 0}$  consider the system

$$\dot{x} = Ax + Bf(x) + Bg(x)m(t) + Fd(t) \quad (8a)$$

$$y = Cx + Gd(t). \quad (8b)$$

Thus, the synchronization problem under the standing assumption and the previous observation is assimilated to the problem of designing an unknown-input observer for (8) that ensures (7). The advantage is that the functions  $f$  and  $g$  are globally Lipschitz. Therefore, it suffices to define  $\Omega$  as the smallest set containing the solutions of (1) for all  $t$ . In general,  $\Omega$  depends on  $r$  i.e., on the ball of initial states  $B_r$  defined previously however, for certain physical systems such as chaotic oscillators,  $\Omega$  is independent of  $r$  as it may be chosen as an arbitrary compact containing the attractor.

### III. ROBUST ADAPTIVE OBSERVER

#### A. State Estimation and Disturbance Rejection

We present an unknown-input adaptive observer that achieves the synchronization objective (7) in presence of external disturbances, parametric uncertainty (piece-wise constant) and measurement noise. The observer is given by the equations

$$\begin{aligned}\dot{z} &= Nz + Jy + Hf(\hat{x}) + Hg(\hat{x})\hat{m} \\ &\quad + \frac{1}{2}\hat{\beta}HM(Ty - C_1\hat{x})\end{aligned} \quad (9a)$$

$$\hat{x} = z - Ey \quad (9b)$$

and the adaptation laws

$$\dot{\hat{m}} = \delta g(\hat{x})^T M(Ty - C_1\hat{x}) \quad (10)$$

$$\dot{\hat{\beta}} = \gamma |M(Ty - C_1\hat{x})|^2. \quad (11)$$

The constants  $\delta$  and  $\gamma$  are positive real numbers imposed by the designer. As we illustrate in simulations, their choice is intricately linked to the rate of change of the piece-wise constant signal  $m$ ; roughly, the gains must be so that estimation convergence is achieved between two switches.

The matrices  $N, E, J, H, T, M$  and  $C_1$  are constant, of appropriate dimensions and are left to be determined. They are required to satisfy the following conditions

$$EG = 0 \quad (12)$$

$$PA - NP - JC = 0 \quad (13)$$

$$PF - NEG - JG = 0 \quad (14)$$

$$H = PB \quad (15)$$

$$P = I + EC \quad (16)$$

$$TG = 0 \quad (17)$$

$$TC = C_1. \quad (18)$$

Furthermore, we require that the triple  $(N, H, C_1)$  satisfies the stability conditions

$$N^T Q + QN < 0 \quad (19)$$

$$H^T Q = MC_1 \quad (20)$$

for a positive definite symmetric matrix  $Q$ .

**Proposition 1:** Under the Standing Assumption and conditions (12)–(20) the expression (7) holds for the solutions  $x(\cdot)$  of

(1) and the observer trajectories  $\hat{x}(\cdot)$  of (9). This, for all initial states  $x(0)$  such that  $x(\cdot)$  are bounded and for all initial states  $\hat{x}(0) \in \mathbb{R}^n$ . That is, the synchronization objective is achieved.

*Remark 1:* Expressions (12)–(20) restrict the class of systems for which the proposed method applies. They establish *structural* conditions of detectability. In particular they may hold for a particular chaotic system with respect to an output but not with respect to another since the latter determines the matrix  $C$ . Correspondingly, not any degree of parametric uncertainty may be coped with as this determines the term  $Bg$ . Necessary and sufficient conditions for expressions (12)–(20) are given in Section III-C.

*Remark 2:* The case of modeling errors e.g. when  $F = F_0 + \Delta F_0$ , where  $F_0$  is an estimated distribution matrix and  $\Delta F_0$  is a modelling error, is beyond the scope of this paper. However, beyond mere convergence, in our main result—Theorem 2, we establish the stronger property of uniform asymptotic stability which entails natural robustness to external perturbations such as  $\Delta F_0 d(t)$ . Namely, the so-called local input-to-state stability. Furthermore, one can use the estimated distribution matrix for the design of the observer and use  $H_\infty$  techniques on the residual term  $\Delta F_0 d(t)$ .

*Proof of Proposition 1:* We start by writing the estimation error dynamics in a convenient form. Then, we use standard Lyapunov arguments to show convergence to zero of the state estimation errors.

To that end define the error variables  $e := x - \hat{x}$  and  $\tilde{m} := m - \hat{m}$ . Hence

$$\begin{aligned} e &= x - z + Ey && \Leftarrow (11b) \\ &= x - z + ECx + EGd && \Leftarrow (1b) \\ &= Px - z && \Leftarrow (14), (18). \end{aligned}$$

Differentiating on both sides of the latter we obtain

$$\begin{aligned} \dot{e} &= PAx + PBf(x) + PBg(x)m(t) + PFd(t) - Nz - Jy \\ &\quad - Hf(\hat{x}) - Hg(\hat{x})\hat{m}(t) - \frac{1}{2}\hat{\beta}HM(Ty - C_1\hat{x}). \end{aligned}$$

Observe that in view of (17) and (18) we have  $Ty - C_1\hat{x} = C_1e$ . Using this, (8b) and  $z = x - e + E(Cx + Gd)$  we obtain

$$\begin{aligned} \dot{e} &= (PA - JC)x - JGd(t) - Nx + Ne - NECx \\ &\quad - NEGd(t) + PBf(x) + PBg(x)m(t) + PFd(t) \\ &\quad - Hf(\hat{x}) - Hg(\hat{x})\hat{m} - \frac{1}{2}\hat{\beta}HMC_1e. \end{aligned}$$

Next, we regroup terms and observe from (14) that the factors of  $d$  equal to zero hence,

$$\begin{aligned} \dot{e} &= (PA - JC - NEC)x - N\hat{x} - \frac{1}{2}\hat{\beta}HMC_1e \\ &\quad + PB[f(x) + g(x)m(t)] - H[f(\hat{x}) + g(\hat{x})\hat{m}]. \end{aligned}$$

From (16) it follows that the (13) is equivalent to  $PA - NEC - JC = N$  therefore, using (15) we obtain

$$\begin{aligned} \dot{e} &= Ne + H[f(x) - f(\hat{x})] + H[g(x)m(t) - g(\hat{x})\hat{m}] \\ &\quad - \frac{1}{2}\hat{\beta}HMC_1e. \quad (21) \end{aligned}$$

Let  $\beta > 0$  be a constant to be determined and define  $\tilde{\beta} := \beta - \hat{\beta}$ . Adding  $Hg(\hat{x})m(t)$  to both sides of (21) we obtain

$$\begin{aligned} \dot{e} &= Ne + H[f(x) - f(\hat{x})] + H[g(x) - g(\hat{x})]m(t) \\ &\quad - \frac{1}{2}\beta HMC_1e + \left[ Hg(\hat{x})\frac{1}{2}HMC_1e \right] \begin{bmatrix} \tilde{m} \\ \tilde{\beta} \end{bmatrix} \quad (22) \end{aligned}$$

where, according to the adaptation laws (10) and (11)  $\tilde{m}$  and  $\tilde{\beta}$  are solutions to<sup>2</sup>

$$\dot{\tilde{m}} = -\delta g(\hat{x})^T MC_1e + \dot{\tilde{m}} \quad \text{a.e.} \quad (23)$$

$$\dot{\tilde{\beta}} = -\gamma |MC_1e|^2. \quad (24)$$

Next, consider the positive definite and radially unbounded function

$$V_1(e, \tilde{\beta}, \tilde{m}) := e^T Qe + \frac{1}{\delta} \tilde{m}^2 + \frac{1}{2\gamma} \tilde{\beta}^2. \quad (25)$$

Its total derivative along the trajectories of (22)–(24) yields<sup>3</sup>

$$\begin{aligned} \dot{V}_1 &= e^T [N^T Q + QN]e + 2e^T QH[f(x) - f(\hat{x})] \\ &\quad - e^T Q[HMC_1e]\beta + 2e^T QH[g(x) - g(\hat{x})]m(t) \\ &\quad + 2e^T QH\tilde{m}g(\hat{x}) - \frac{2}{\delta} \tilde{m}^T [\delta g(\hat{x})^T MC_1e - \dot{\tilde{m}}(t)] \end{aligned}$$

where we have used (20). Next, let condition (19) generate a positive real constant  $\eta$  such that  $N^T Q + QN \leq -2\eta$ . Using Fact 1.2 with  $w^T = e^T QH$ , (20) and (2) we obtain

$$\begin{aligned} \dot{V}_1 &\leq -2\eta|e|^2 + 2|MC_1e|[K_f + K_g K_m]|e| \\ &\quad - \beta|MC_1e|^2 - \frac{2}{\delta} \tilde{m}^T \dot{\tilde{m}}(t). \end{aligned}$$

Let  $\beta := [K_f + K_g K_m]^2 / \eta$  then, using the triangle inequality on  $2|MC_1e|[K_f + K_g K_m]|e|$  we finally obtain

$$\dot{V}_1(e, \tilde{m}, \tilde{\beta}) \leq -\eta|e|^2 - \frac{2}{\delta} \tilde{m}^T \dot{\tilde{m}}(t)$$

which holds for all  $e$  such that  $x \in \Omega$  and almost all  $t \geq 0$ . That is, for any compact  $\Omega$  and for all initial conditions  $x_0 \in \mathbb{R}^n$  such that the solutions  $x(\cdot)$  remain in the set  $\Omega$  and for almost all  $t$  we have

$$\dot{V}_1(e(t), \tilde{m}(t), \tilde{\beta}(t)) \leq -\eta|e(t)|^2 - \frac{2}{\delta} \tilde{m}(t)^T \dot{\tilde{m}}(t)$$

which, using the assumption that  $\dot{\tilde{m}}(t) = 0$  almost everywhere, implies that

$$\dot{V}_1(e(t), \tilde{m}(t), \tilde{\beta}(t)) \leq -\eta|e(t)|^2 \quad \text{a.e.} \quad (26)$$

Integrating on both sides of the latter from 0 to  $\infty$  we obtain that  $e(t)$ ,  $\tilde{m}(t)$  and  $\tilde{\beta}(t)$  are bounded for all  $t$  and moreover,  $e(t)$  is square integrable. Next, we make the following substitutions in (22):

$$\begin{aligned} x &= x(t) & \hat{x} &= x(t) - e(t) & \dot{e} &= \dot{e}(t) \\ \tilde{m} &= \tilde{m}(t) & \tilde{\beta} &= \tilde{\beta}(t) \end{aligned}$$

to conclude that  $\dot{e}(t)$  is also uniformly bounded for all  $t$  (since  $m(t)$  is bounded). By [25, Lemma A.5] we conclude that

$$\limsup_{t \geq 0} |e(t)| = 0$$

that is, (7) holds.  $\square$

<sup>2</sup>Note that since  $m$  is piece-wise constant  $\dot{m}(t) = 0$  for almost all  $t$ .

<sup>3</sup>Strictly speaking, this is valid, except for the points where  $\dot{m}$  does not exist which is a countable number of points.

### B. Parametric Convergence

It has been mentioned that the literature lacks from a strict analysis on parametric convergence—see [15]–[17]. However, in the articles [21], [26] an extensive treatment of this issue is presented through different approaches: tracking control and observer-based respectively. We find it fit to shortly discuss the fundamental theory that is used to establish parametric convergence for non-autonomous nonlinear systems, which cover chaotic systems.

Proposition 1 establishes the convergence of the estimation errors to zero. However, in general this is a weak property (it does not imply robustness). We show now that under the conditions of the proposition and an additional assumption of *persistence of excitation*, uniform global asymptotic stability of the origin of the error system i.e.,  $(e, \tilde{m}, \tilde{\beta}) = (0, 0, 0)$  may be obtained. Moreover, the excitation assumption is necessary and sufficient.

Roughly speaking, persistency of excitation is a concept of average which, under *appropriate* structural hypotheses leads to the statement of necessary and sufficient conditions for asymptotic stability and parameter convergence. It seems fitting to briefly discuss this property as it is fundamental to our main results.

Consider the pulse train function  $a(t)$  defined as  $a = 1$  if  $t \in (0, 1] \cup (2i, 2i + 1]$  and  $a = 0$  if  $t \in (2i - 1, 2i]$  with  $i \in \mathbb{N}$ . This function possesses the property of persistency of excitation: there exist<sup>4</sup>  $T > 0$  and  $\mu > 0$  such that

$$\int_t^{t+T} |a(s)| ds \geq \mu \quad \forall t \geq 0.$$

Then, from the literature on stability of linear systems and adaptive control we know that the origin  $(e, \theta) = (0, 0)$  of the system

$$\dot{e} = -e + a(t)e\theta \quad (27a)$$

$$\dot{\theta} = -a(t)e \quad (27b)$$

is exponentially stable. This implies the convergence of the parametric error  $e_\theta \rightarrow 0$ .

Persistency of excitation is omnipresent in Model Reference Adaptive Control—[27], [28] of linear systems. Yet, the closed-loop system is typically *nonlinear*. For the purpose of mere illustration, consider the system

$$\dot{e} = -e + a(t)e^2e_\theta \quad (28a)$$

$$\dot{e}_\theta = -a(t)e^2. \quad (28b)$$

If we intend to apply tools tailored for linear systems to conclude parametric convergence we would naturally impose that

$$\int_t^{t+T} |a(s)e^2| ds \geq \mu \quad \forall t \geq 0.$$

For notational simplicity, arguments are often omitted in the literature, leading to fundamental mistakes. In the integral above, it is crucial to define what  $e$  is. If we consider that it is the state

of (28) then, the correct manner to write the condition seems to be

$$\int_t^{t+T} |a(s)e(s)^2| ds \geq \mu \quad \forall t \geq 0.$$

The latter has two major drawbacks as the integral is computed along system's *trajectories*:

- 1) as  $e \rightarrow 0$  obviously the persistency of excitation is lost;
- 2) it is implicitly implied that the error trajectories  $e(t)$  are known ahead of time in order to evaluate the integral for all time.

Note that the stabilization goal is precisely to steer the error  $e \rightarrow 0$  hence, in view of 1) above, the necessary condition for stabilization may seemingly not be met.

The following is a *relaxed* notion of persistency of excitation tailored for functions that depend on time and state variables. However, the following definition is not a trajectory-based property and leads to considerably relaxed stability conditions. Let  $z = [z_1, z_2]$  be a state variable and consider a uniformly continuous function  $\phi(\cdot, \cdot)$ . It is uniformly  $\delta$  persistently exciting with respect to  $z_1$  if and only if there exist  $T > 0$  and  $\mu > 0$  such that, for all  $t \in \mathbb{R}$

$$|z_1| \neq 0 \implies \int_t^{t+T} |\phi(\tau, z)| d\tau \geq \mu.$$

Notice that for the system (28) we can define the regressor function  $\phi(t, z) = a(t)e^2$  and  $z_1 = e$  and  $z_2 = e_\theta$ . Then, the function  $a(t)e^2$  is uniformly  $\delta$ -persistently, with respect to  $z_1$  because

$$|e| \neq 0 \implies \int_t^{t+T} |a(\tau)e^2| d\tau \geq \mu e^2 > 0.$$

In the integral above  $e$  is a fixed variable and not a state trajectory. Note that the condition to be verified holds if and only if  $a$  is persistently exciting. Then, it is possible to establish that  $(e, e_\theta) = (0, 0)$  is exponentially stable for (28).

*Remark 3:* The following observations are in order.

- 1) As mentioned in the Introduction, although it has been recognized a lack of strict analysis in the literature references such as [15], [16], [20] present interesting and highly intuitive arguments to establish parametric convergence. In [15], [16] the sufficient conditions may roughly be stated for the system above as demanding that the function  $a(t)e^2$  is linearly independent from  $e_\theta$ . This is tantamount to ask that  $|a(t)e^2e_\theta| > 0$  for all  $e_\theta \neq 0$  which obviously cannot hold, in general—even if  $e \neq 0$  the function  $a(t) = 0$  for certain  $t$ . However, it is sufficient that linear independence holds over a sufficiently long interval starting at any time. This is the rationale behind persistency of excitation.
- 2) Also, note that there is no contradiction with the argument in the boxed paragraph above. Indeed, what is clear is that persistency of excitation as originally defined for functions of time only and tailored to be used in stability theorems for linear systems, may not be used in the analysis of nonlinear

<sup>4</sup>In this example,  $T = 2$  and  $\mu = 1$ .

systems. The general statement for nonlinear time-varying systems which we present next is used to prove our main result. The reader is kindly invited to see [26], [29] for further discussions and illustration on the use of persistency of excitation for chaotic systems.  $\square$

Consider nonlinear time-varying systems  $\dot{z} = F(t, z)$  with

$$F(t, z) := \begin{bmatrix} A(t, z) + B(t, z) \\ C(t, z) \end{bmatrix} \quad (29)$$

for which it is assumed that  $A(\cdot, 0) \equiv 0$ ,  $B(\cdot, 0) \equiv 0$  and  $C(\cdot, 0) \equiv 0$ . Furthermore, define

$$B_o(t, z_2) := B(t, z)|_{z_1=0} \quad (30)$$

and notice that necessarily,  $B_o(\cdot, 0) \equiv 0$ . Let the following hypotheses hold.

*Assumption 1:* There exists a locally Lipschitz function  $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ , class- $\mathcal{K}_\infty$  functions  $\alpha_1, \alpha_2$  and a continuous, positive definite function  $\alpha_3$  such that

$$\alpha_1(|z|) \leq V(z, t) \leq \alpha_2(|z|) \quad (31)$$

and, almost everywhere

$$\dot{V}(z, t) \leq -\alpha_3(z_1). \quad (32)$$

*Assumption 2:* The functions  $A$ ,  $B$  and  $C$  are locally Lipschitz in  $z$  uniformly in  $t$ . Moreover, for each  $\Delta \geq 0$  there exist  $b_M > 0$  and continuous nondecreasing functions  $\rho_i : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  such that  $\rho_i(0) = 0$  and for almost all  $t \in \mathbb{R}$  and  $z \in \mathbb{R}^n$

$$\max_{|z_2| \leq \Delta} \left\{ |B_o(t, z_2)|, \left| \frac{\partial B_o}{\partial t} \right|, \left| \frac{\partial B_o}{\partial z_2} \right| \right\} \leq b_M \quad (33)$$

$$|B(t, z) - B_o(t, z_2)| \leq \rho_1(|z_1|) \quad (34)$$

$$\max_{|z_2| \leq \Delta} \{|A(t, z)|, |C(t, z)|\} \leq \rho_2(|z_1|). \quad (35)$$

*Theorem 1:* The system (29) under Assumptions 1–2 is uniformly globally asymptotically stable if and only if  $B_o(\cdot, \cdot)$  is uniformly  $\delta$ -persistently exciting w.r.t.  $z_2$ .

We are ready to present our main result for the adaptive observer (9).

*Theorem 2:* Consider the system (1) and the observer (9) with the adaptation laws (10), (11). Let the Standing Assumption and conditions (12)–(20) hold. Then, the origin  $(e, \tilde{m}, \tilde{\beta}) = (0, 0, 0)$  of the error system, (22)–(24) is uniformly globally asymptotically stable if and only if there exist  $\mu, T > 0$  such that

$$\int_t^{t+T} g(x(s))^\top H^\top H g(x(s)) ds \geq \mu \quad \forall t \geq 0. \quad (36)$$

*Remark 4:* Notice that persistency of excitation is assumed to hold for the function  $g$  along the master trajectories only and not along *error* trajectories which are meant to converge to zero. This is a crucial since the master system is meant to remain in *chaotic* regime, by design. That is,  $x(s)$  in (36) is not steered to zero and it may be safely assumed that (36) holds.

*Proof:* The proof follows invoking Theorem 1 with the following definitions:  $z_1 = e$ ,  $z_2 = \text{col}[\tilde{m}, \tilde{\beta}]$ , hence using  $\hat{x}(t) = x(t) - z_1$

$$\begin{aligned} A(t, z) &:= Ne + H[f(x(t)) - f(x(t) - z_1)] \\ &\quad + H[g(x(t)) - g(x(t) - z_1)]m(t) \\ &\quad - \frac{1}{2}\beta HMC_1 z_1 \\ B(t, z) &:= \begin{bmatrix} Hg(x(t) - z_1) & \frac{1}{2}HMC_1 z_1 \end{bmatrix} \begin{bmatrix} \tilde{m} \\ \tilde{\beta} \end{bmatrix} \\ C(t, z) &:= \begin{bmatrix} -\delta g(x(t) - z_1)^\top MC_1 z_1 - \dot{m} \\ -\gamma |MC_1 z_1|^2 \end{bmatrix}. \end{aligned}$$

Clearly, Assumption 2 of Theorem 1 holds. The necessary and sufficient condition of persistency of excitation is equivalent to (36) since  $B_o(t, z_2) = Hg(x(t))\tilde{m}$ . That Assumption 1 holds follows from the proof of Proposition 1.  $\blacksquare$

### C. Implementation

We have showed that the slave system (9) achieves the synchronization goal under conditions (12)–(20). In this section we show in detail how these conditions may be verified. In particular, we present a procedure to determine the design matrices involved in (12)–(20).

Consider first (17) and (18). They may be re-written as

$$\begin{bmatrix} T & T \end{bmatrix} \begin{bmatrix} C & 0 & I \\ 0 & G & -I \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} \quad (37)$$

which is of the form

$$XR_1 = R_2 \quad (38)$$

in which, given  $R_1$  and  $R_2$ , it is required to find  $X$ . After [23] (38) is solvable if and only if

$$\text{Rank} \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \text{Rank}[R_1] \quad (39)$$

and the solution is given, for any  $Z$ , by

$$X = R_2 R_1^+ - Z(I - R_1 R_1^+) \quad (40)$$

where  $R^+$  denotes the generalized inverse of  $R$  i.e., such that<sup>5</sup>  $RR^+R = R$ . Thus,  $T$  may be obtained from (40) with

$$X = \begin{bmatrix} T & T \end{bmatrix}, \quad R_1 = \begin{bmatrix} C & 0 & I \\ 0 & G & -I \end{bmatrix}, \quad R_2 = \begin{bmatrix} C_1 & 0 & 0 \end{bmatrix} \quad (41)$$

Consider next, (13) and (14). Given an arbitrary matrix  $K$  let  $J = -NE - K$  so (13) and (14) become, respectively

$$PA + KC = N \quad (42)$$

$$KG = -PF. \quad (43)$$

<sup>5</sup>A practical method to compute  $R^+$ , is to find a singular value decomposition of  $R$  such that  $R = USV^\top$  where  $U$  and  $V$  are unitary matrices and e.g.,  $S := [S_1 \ 0]$  non-negative and of the same dimension than  $R$ , with  $S_1$  square and diagonal. Hence,  $R^+ = V\tilde{S}U^\top$  where  $\tilde{S} := [S_1^{-1} \ 0]^\top$ . By definition,  $R^+ = 0$  if  $R = 0$ .

The necessary and sufficient condition for (43) to be solvable for  $K$  is that

$$\text{Rank} \begin{bmatrix} G \\ -PF \end{bmatrix} = \text{Rank}[G] \quad (44)$$

and, according to (40) its general solution is

$$K = -PFG^+ - L(I - GG^+) \quad (45)$$

where  $G^+$  is a generalized inverse of  $G$  and  $L$  is an arbitrary matrix of appropriate dimensions. Using (45) in (42), we obtain

$$N = A_1 - LC_1 \quad (46)$$

where

$$A_1 = PA - PFG^+C \quad (47)$$

$$C_1 = (I - GG^+)C. \quad (48)$$

After [30], the necessary and sufficient conditions for the solvability of (19) and (20) are

$$\text{Rank}[C_1H] = \text{Rank}[H] \quad (49)$$

$$\text{Rank} \begin{bmatrix} A_1 - \lambda I & H \\ C_1 & 0 \end{bmatrix} = n + \text{Rank}[H] \quad (50)$$

for each complex number  $\lambda$  such that  $\text{Re}(\lambda) \geq 0$ . To solve (19), (20) with  $N = A_1 - LC_1$  we consider the following convex optimization problem—cf. [30]: to minimize  $\rho$  subject to

$$Q > 0 \quad (51)$$

$$QA_1 + A_1^\top Q + WC_1 + C_1^\top W^\top < 0 \quad (52)$$

$$\begin{bmatrix} \rho I & H^\top Q - MC_1^\top \\ QH - C_1^\top M^\top & \rho I \end{bmatrix} \leq 0. \quad (53)$$

The solution to the latter yields the minimum  $\rho = 0$  and  $Q, M$  such that  $L = -Q^{-1}W$  satisfies (19), (20) with  $N = A_1 - LC_1$ . ■

We have showed that the adaptive observer (9)–(11) achieves estimation error convergence for a fairly general class of non-linear systems. The necessary and sufficient conditions are given by the rank conditions (39), (44), (49) and (50) which impose structural properties on the system.

We conclude the section with the following simple algorithm to compute the design matrices.

- Step 1: Compute  $C_1$  using (48);
- Step 2: compute  $T$  using (40) and (41);
- Step 3: if  $p = n$ , choose  $E = T$ . Else, if  $p < n$ , choose  $E = \begin{bmatrix} T \\ 0 \end{bmatrix}$ ;
- Step 4: compute  $H$  and  $P$  using respectively (15) and (16);
- Step 5: compute  $A_1$  using (47);
- Step 6: find matrices  $M, L$  and  $Q$  by solving the convex optimization problem, presented in the above analysis. We can also apply the detailed method proposed in [30], to determine  $M, L$  and  $Q$ .
- Step 7: Compute  $K$  and  $N$  using respectively (45) and (46);
- Step 8: compute  $J = -NE - K$ .

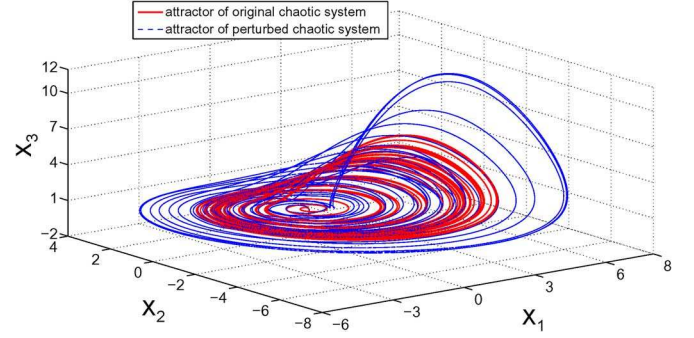


Fig. 1. Attractors of the chaotic Rössler system (4) and the Rössler-based transmitter (54).

#### IV. CHAOTIC-BASED SYNCHRONIZATION FOR TRANSMISSION OF INFORMATION

We present now two examples on how to apply our synchronization approach in tasks of transmission of information using chaotic carriers. Indeed, the development of chaotic synchronization has been long steered by the aim at enabling telecommunication robust to attacks i.e., interception of transmitted information. See for instance [2], [31]. However, the technique must be used with caution as it is not universally robust to attacks. For instance, [32] shows particular examples for which parameter-modulation based, master-slave synchronization fails in masking the information; indeed, the latter may be recovered using simple technology (Butterworth filters, etc.). See also [14].

##### A. Example 1: Rössler Systems

Consider a Rössler-based transmitter circuit with  $a = 0.398$ ,  $b = 2$  and  $c = 4$ . With this choice of parameters the system (4) has a chaotic behavior. We assume that the measured outputs are  $y = [y_1, y_2]^\top$  as previously defined. We assume that the same perturbing input  $d(t)$  acts on the three dynamic equations. Besides, a digital message  $m(t)$  is injected in the system. That is, the dynamics of the transmitter is given by

$$\dot{x}_1 = -(x_2 + x_3) + d(t) \quad (54a)$$

$$\dot{x}_2 = x_1 + ax_2 + d(t) \quad (54b)$$

$$\dot{x}_3 = b + x_3(x_1 - c) + m(t)x_3 + d(t) \quad (54c)$$

$$y_1 = x_1 + 2d(t) \quad (54d)$$

$$y_2 = x_3 + d(t) \quad (54e)$$

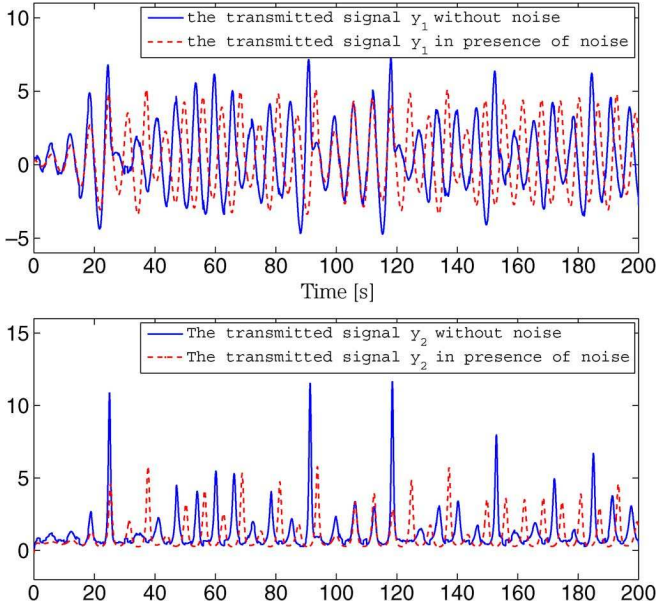
which is of the form (1) with  $g_0(x) = x_3$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Under these conditions the transmitter (54) exhibits chaotic behavior i.e., in spite of the additive disturbances and the injection of the information signal. For comparison, Fig. 1 depicts the attractors of systems (54) and (4) with the initial conditions set to  $x_0 = [0.2, -0.4, -0.2]^\top$  for both systems. The effect of noise in the measurements is appreciated in Fig. 2.

Let  $\Omega := \{x \in \mathbb{R}^3 : |x_i| \leq \omega_i, \forall i \in \{1 \dots 3\}\}$ . Note from Fig. 1 that  $\Omega$  strictly contains the systems' attractors. Hence, the saturation levels are set to  $\omega_1 = \omega_2 = \omega_3 = 12$  and we define



Fig. 2. Graph of measured outputs  $y_1(t)$  and  $y_2(t)$ .

$f(x) := f_0(\sigma(x))$ ,  $g(x) = g_0(\sigma(x))$ . Clearly,  $f(x) = f_0(x)$  and  $g(x) = g_0(x)$  for all  $x \in \Omega$ . Hence, the transmitter (54) takes the form (8) with  $f(x) = 2 + \sigma_1(x)\sigma_3(x)$ ,  $g(x) = \sigma_3(x)$ .

The receiver system is given by the system (9) and the adaptation laws (10) and (11). To find explicit numeric values of the observer matrices we follow the tuning procedure previously described in Section III-C. We obtain

$$\begin{aligned}
 T &= \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix}, \quad E = \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \\ 0 & 0 \end{bmatrix} \\
 H &= \begin{bmatrix} -0.4 \\ 0.8 \\ 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1.2 & 0 & -0.4 \\ -0.4 & 1 & 0.8 \\ 0 & 0 & 1 \end{bmatrix} \\
 A_1 &= \begin{bmatrix} -0.32 & -1.2 & 0.24 \\ 0.44 & 0.798 & -3.08 \\ -0.4 & 0 & -4.2 \end{bmatrix} \\
 C_1 &= \begin{bmatrix} 0.2 & 0 & -0.4 \\ -0.4 & 0 & 0.8 \end{bmatrix} \\
 Q &= \begin{bmatrix} 5.1283 & 4.9395 & -2.6505 \\ 4.9395 & 6.5315 & -3.2496 \\ -2.6505 & -3.2496 & 3.0395 \end{bmatrix} \\
 L &= \begin{bmatrix} 19.9972 & -39.9943 \\ -40.0010 & 80.0020 \\ -50.0030 & 100.0059 \end{bmatrix}, \quad M^\top = \begin{bmatrix} -0.75 \\ 1.5 \end{bmatrix} \\
 K &= \begin{bmatrix} -20.3172 & 39.8343 \\ 39.441 & -80.282 \\ 49.603 & -100.2059 \end{bmatrix} \\
 N &= \begin{bmatrix} -20.3172 & -1.2 & 40.2343 \\ 40.4410 & 0.798 & -83.082 \\ 49.603 & 0 & -104.2059 \end{bmatrix} \\
 J &= \begin{bmatrix} 23.9006 & -47.0012 \\ -47.21 & 95.82 \\ -59.5235 & 120.0471 \end{bmatrix}.
 \end{aligned}$$

The adaptation gains are  $\delta = 200$  and  $\gamma = 5000$ .

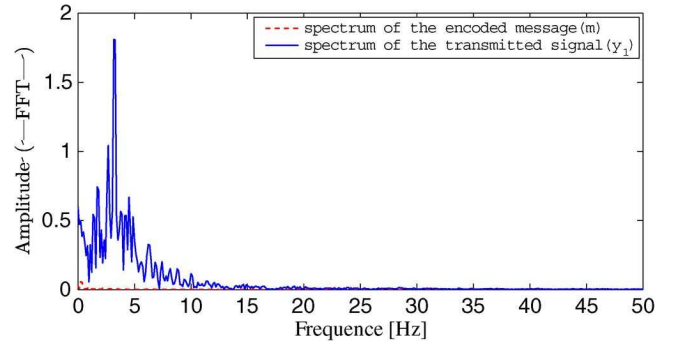


Fig. 3. Spectrum diagram of the carrier signal and the information.

To put our contribution in perspective, albeit through this particular example, we have run simulations using the adaptive observer (9) as well as the “Luenberger-type” adaptive observer

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + Bg(\hat{x})\hat{m} + L_2(y - C\hat{x}) \quad (55a)$$

$$\dot{\hat{m}} = kg(\hat{x})^\top M_2(y - C\hat{x}) \quad (55b)$$

where  $k$  is a positive number—cf. [7], [21], [30]. Referring to the latter two we use

$$L_2 = \begin{bmatrix} 0.8424 & 0 \\ -0.6459 & 0 \\ 0 & 20 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and  $k = 200$ .

We assume that the message to transmit is a digital pulse train emulated by  $m(t) = 0.5\text{sgn}(\sin(0.2t))$ . Fig. 3 depicts the frequency spectrum of the message and the carrier signals.

The initial conditions for both decoders, (9) and (55) are set to  $\hat{x}_0 = [-0.12, 0.24, 0]^\top$  and the estimated message  $\hat{m}$  is initialized at  $\hat{m}_0 = 0$ . That is, the two slave systems (9) and (55) are driven by the same master system (4), submitted to the same initial conditions with aim at recovering the signal  $m(\cdot)$ .

The simulations are as follows. Firstly tests are carried out without noise (i.e., with  $d = 0$ ); the obtained results are depicted in Figs. 4, 5, 6 and 7. Note that the two receivers have an acceptable performance in terms of information recovery and estimation error. However, both transient performance and asymptotic behavior of the adaptive observer (9) supersedes that of (55) which in particular, presents transient oscillations and a steady-state estimation error. This is not surprising for such a simple scheme, as is formally studied in [14].

In a second run of simulations the transmitter system (54) is perturbed. The perturbation function  $d$  consists in an uniformly distributed noise (random) signal generated between lower and upper bounds respectively equal to 0 and 0.4. This corresponds to a S/N ratio of  $-17.5$  dB.

The simulation results are showed in Figs. 8 and 9. Note in the *former* that the effect of noise and disturbances is perfectly canceled out by the adaptation laws (10) and (11) and the tuning procedure from Section III-C. Indeed, the synchronization error is not affected—cf. Fig. 8. For comparison, we show in Fig. 9, the performance of the alternative receiver (55).

The performance improvement in the presence of noise and disturbances is clearer from Figs. 10 and 11 in which we show the transmitted and recovered signals for both observers (9) and (55) respectively.

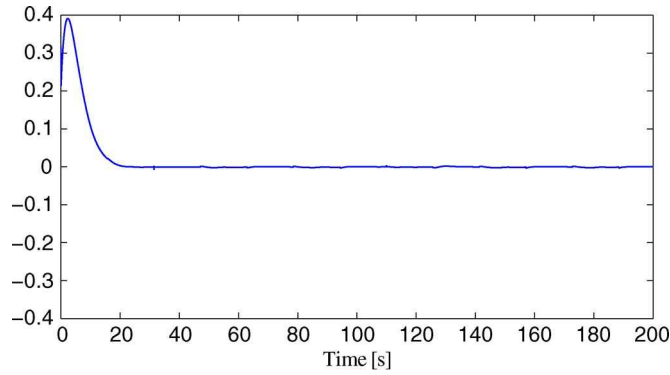


Fig. 4. Synchronization error  $e_1 = x_1 - \hat{x}_1$  for the adaptive robust observer (9) in the absence of noise and disturbances.

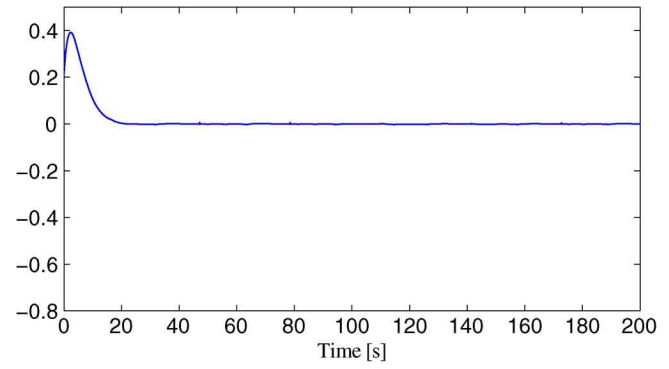


Fig. 8. Synchronization error  $e_1 = x_1 - \hat{x}_1$  using observer (9) in presence of noise and disturbances.

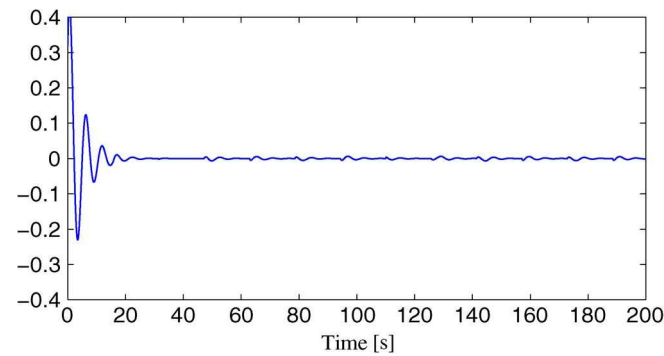


Fig. 5. Synchronization error  $e_1 = x_1 - \hat{x}_1$  by applying the observer (55) in the absence of noise and disturbances.

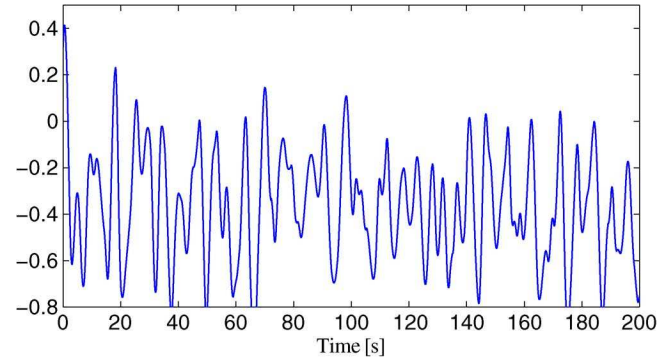


Fig. 9. Synchronization error  $e_1 = x_1 - \hat{x}_1$  using the observer (55) in presence of noise and disturbances.

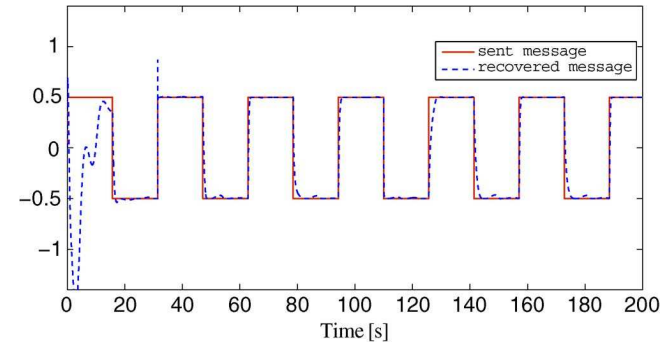


Fig. 6. The transmitted message  $m$  and the recovered message  $\hat{m}$  by the adaptive observer (9) in the absence of noise and disturbances.

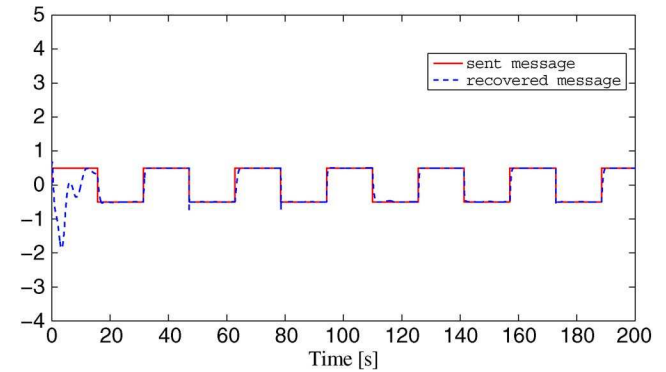


Fig. 10. The transmitted message  $m$  and the recovered message  $\hat{m}$  by applying observer (9) in the presence of noise and disturbances.

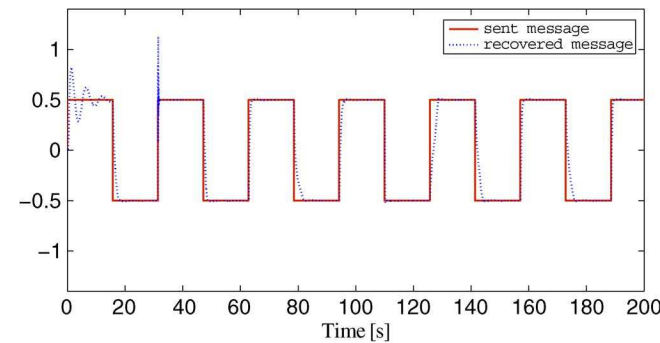


Fig. 7. The transmitted message  $m$  and the recovered message  $\hat{m}$  by applying the observer (55) in the absence of noise and disturbances.

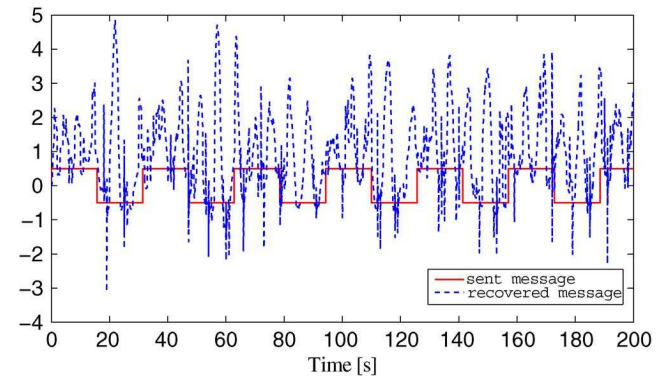


Fig. 11. The transmitted message  $m$  and the recovered message  $\hat{m}$  by applying observer (55) in the presence of noise and disturbances.

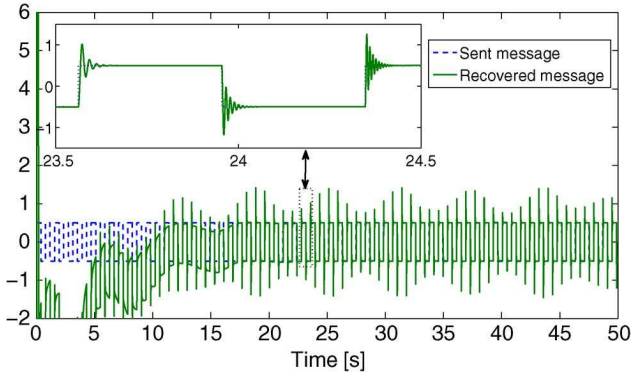


Fig. 12. The transmitted high-frequency message  $m$  and the recovered message  $\hat{m}$  by applying observer (9)–(11) in the presence of noise and disturbances; “high” adaptation gain.

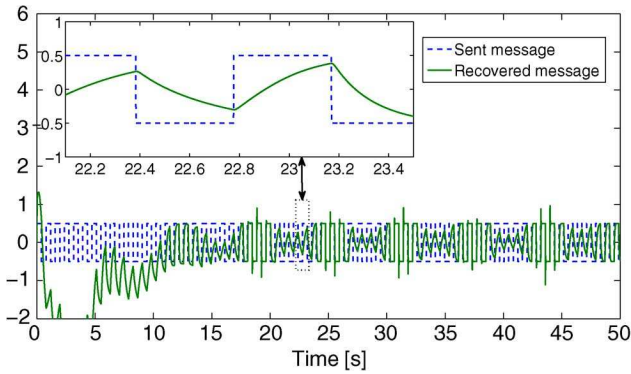


Fig. 13. The transmitted high-frequency message  $m$  and the recovered message  $\hat{m}$  by applying observer (9)–(11) in the presence of noise and disturbances; “low” adaptation gain.

In the previous simulations the frequency of the train pulse used to mimic the information i.e.,  $m(t)$  is rather low. Other tests were performed with higher switching frequency of the piece-wise constant function  $m(t)$ . Figs. 12 and 13 depict the estimation of a message with a relatively fast switching frequency and by applying respectively, two different adaptive gains ( $\delta_1 = 20000$  and  $\delta_2 = 5000$ ). As it is showed, in the case of fast switching frequencies, we must increase sufficiently the adaptive gain in order to improve the quality of the recovered message. We also see that the larger is the adaptive gain, the faster is the convergence rate of the parameter estimation error (i.e., the lower is the convergence time).

### B. Example 2: Genesio-Tesi System With Uncertainties

We present another case-study, in which besides the unknown input (the coded message) it is assumed that the model contains parameter uncertainties. The latter are estimated via an adaptation law, besides recovering the unknown input. The master system consists of a Genesio-Tesi chaotic system subject to perturbations both in the dynamics and in the output equations i.e.,

$$\dot{x}_1 = x_2 + d(t) \quad (56a)$$

$$\dot{x}_2 = x_3 + d(t) \quad (56b)$$

$$\dot{x}_3 = -[c + m_1(t)]x_1 - bx_2 - ax_3 + x_1^2 + d(t) \quad (56c)$$

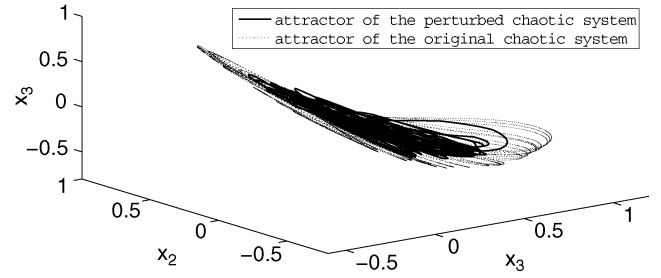


Fig. 14. Attractors of the chaotic system (56) with and without perturbations.

where  $d$  is a (bounded) disturbance. The measured outputs are

$$y_1 = x_1 + 2d(t), \quad y_2 = x_1 + x_3 + d(t). \quad (57)$$

According to the previous developments, we re-write the system in the form of (58) i.e.,

$$\begin{cases} \dot{x} = Ax + Bf_0(x) + Bg_0(x)m + Fd \\ y = Cx + Gd \end{cases} \quad (58)$$

where  $x = [x_1, x_2, x_3]^T$ ,  $y = [y_1, y_2]^T$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$G = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad m = \begin{bmatrix} a \\ b \\ m_1 \end{bmatrix}$$

$f_0(x) = x_1^2$  and  $g_0(x) = [-x_3, -x_2, -x_1]$ . It is assumed that the parameters  $a$  and  $b$  are unknown. The information signal is  $m_1(t) = 0.05\text{sgn}(\sin(0.05t))$ . We initialize the state of system (56) at  $x_0 = [0.2, -0.4, -0.2]^T$ . Fig. 14 depicts the attractor of system (56) without noise ( $d = 0$ ) and in presence of uniformly distributed noise generated between lower and upper bounds respectively equal to 0 and 0.1 (13). Fig. 16 shows the effect of the additive noise on the transmitted signal  $y_1 = x_1 + 2d$  against time. The signal-to-noise ratio is  $S/N = -14$  dB.

Consider the compact set  $\Omega = \{x \in \mathbb{R}^n : |x_i| \leq \omega_i, 1 \leq i \leq 3\}$  with  $\omega_1 = \omega_2 = \omega_3 = 2$ . We can deduce from Fig. 14, that  $\Omega$  contain strictly the attractor of the system (57). Let  $\sigma(x)$  be a saturation function defined as follows:  $\sigma(x) = [\sigma_1(x), \sigma_2(x), \sigma_3(x)]^T$ , and for  $1 \leq i \leq 3$

$$\sigma_i(x) = \begin{cases} \omega_i & \text{if } x_i > \omega_i \\ x_i & \text{if } -\omega_i \leq x_i \leq \omega_i \\ -\omega_i & \text{if } x_i < -\omega_i. \end{cases} \quad (59)$$

In this case, for each  $x \in \Omega$ ,  $f(x) = f_0(\sigma(x)) = f_0(x)$  and  $g(x) = g_0(\sigma(x)) = g_0(x)$ . Moreover,  $f(x)$  and  $g(x)$  are globally Lipschitz in  $x$  with Lipschitz constants  $K_f$  and  $K_g$  respectively. Then, the system can be written in the form

$$\begin{cases} \dot{x} = Ax + Bf(x) + Bg(x)m + Fd \\ y = Cx + Gd \end{cases} \quad (60)$$

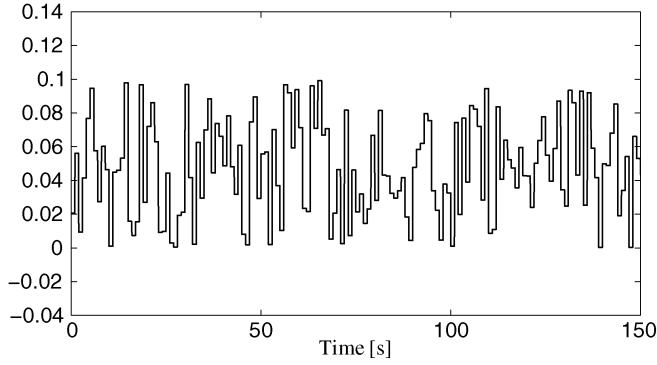
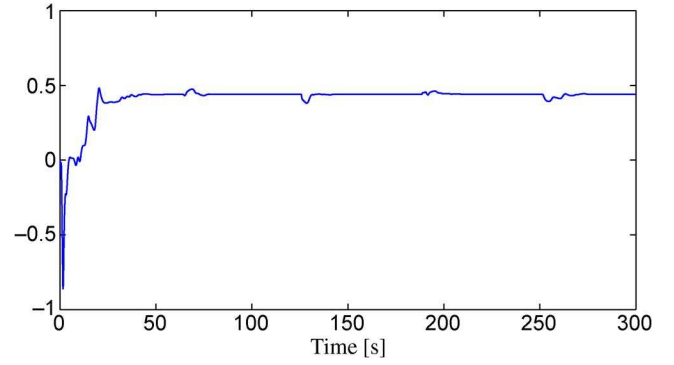
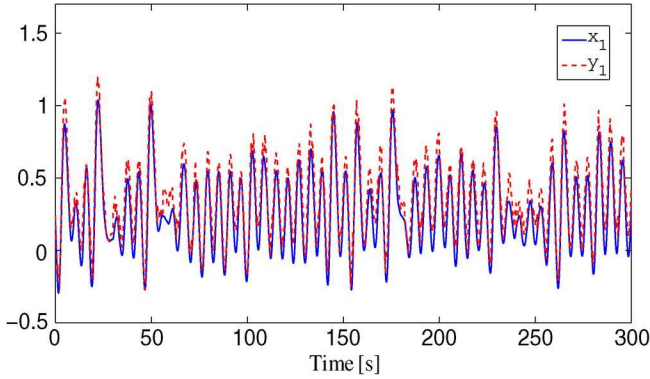
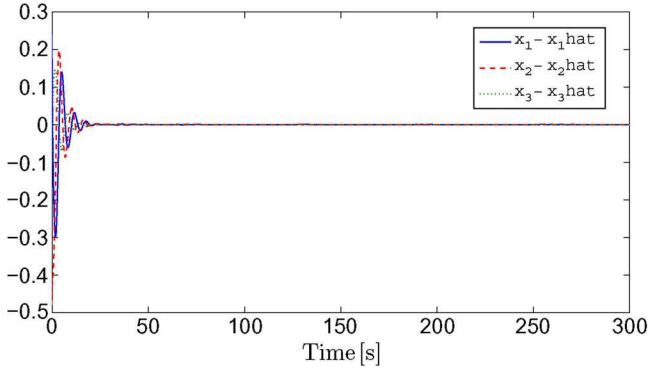
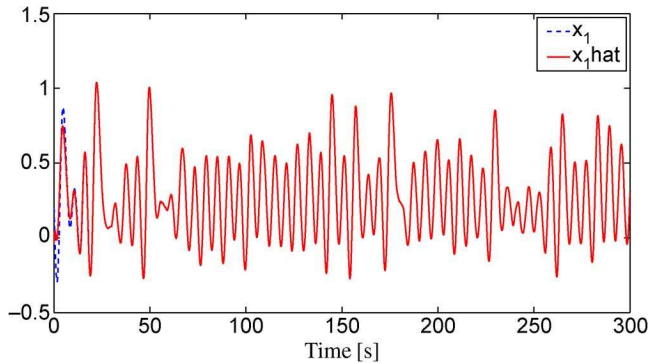


Fig. 15. The uniformly distributed noise.

Fig. 19. Estimation of the parameter  $a$ .Fig. 16. Effect of the additive noise on the transmitted signal  $y_1$ .Fig. 17. Synchronization errors  $e_1 = x_1 - \hat{x}_1$ ,  $e_2 = x_2 - \hat{x}_2$  and  $e_3 = x_3 - \hat{x}_3$ .Fig. 18. Comparison between the state  $x_1$  and its estimate  $\hat{x}_1$ .

with  $f(x) = \sigma_1(x)^2$ , and  $g(x) = [-\sigma_3(x), -\sigma_2(x), -\sigma_1(x)]$ . Note that, from the chaotic behavior of the transmitter (60),

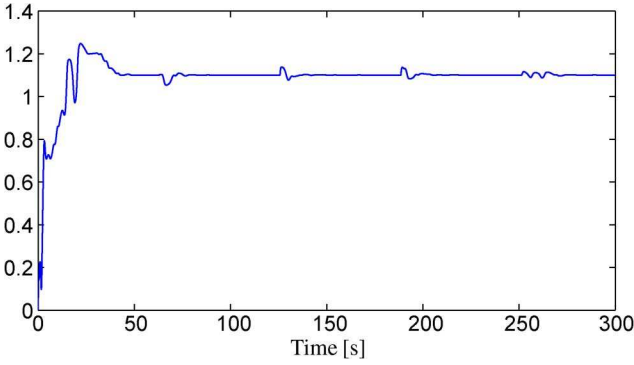
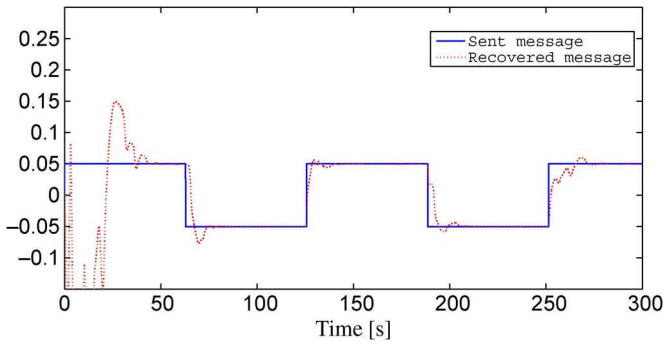
we can deduce that  $[Hg(x(t))]^T$  is persistently exciting, and its time derivative is bounded. The transmitted signal  $y$  drives the adaptive unknown-input observer having the form of the system (9) associated with the adaptation laws (10) and (11) and  $\delta = 150$  and  $\gamma = 5000$ . The observer gains are selected following the algorithm in Section III-C. We obtain

$$\begin{aligned}
 T &= \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \end{bmatrix}, \quad E = \begin{bmatrix} 0.2 & -0.4 \\ -0.4 & 0.8 \\ 0 & 0 \end{bmatrix} \\
 H &= \begin{bmatrix} -0.4 \\ 0.8 \\ 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1.2 & 0 & -0.4 \\ -0.4 & 1 & 0.8 \\ 0 & 0 & 1 \end{bmatrix} \\
 A_1 &= \begin{bmatrix} -0.24 & 0.8 & -0.08 \\ -1.32 & 0.4 & 0.56 \\ -0.6 & 0 & -0.2 \end{bmatrix} \\
 C_1 &= \begin{bmatrix} -0.2 & 0 & -0.4 \\ 0.4 & 0 & 0.8 \end{bmatrix} \\
 Q &= \begin{bmatrix} 9.7684 & -3.4037 & 7.7542 \\ -3.4037 & 4.5589 & -5.0082 \\ 7.7542 & -5.0082 & 9.3571 \end{bmatrix} \\
 L &= \begin{bmatrix} 4.9987 & -9.9973 \\ -9.999 & 19.998 \\ -12.4992 & 24.9984 \end{bmatrix}, \quad M^T = \begin{bmatrix} -1.0629 \\ 2.1258 \end{bmatrix} \\
 K &= \begin{bmatrix} -5.1587 & 9.9173 \\ 9.119 & -20.438 \\ 12.0992 & -25.1984 \end{bmatrix} \\
 N &= \begin{bmatrix} 1.3763 & 0.8 & 9.9173 \\ -4.2530 & 0.4 & -19.438 \\ -3.4198 & 0 & -25.1984 \end{bmatrix} \\
 J &= \begin{bmatrix} 1.4210 & -2.4420 \\ -0.2424 & 2.6848 \\ -0.7359 & 2.4717 \end{bmatrix}.
 \end{aligned}$$

The observer's state is initialized at  $\hat{x}_0 = [-0.12, 0.24, 0]^T$  and the estimated parameter vector  $\hat{m}$  is initialized as  $\hat{m}_0 = 0$ .

Simulation results depicted in Figs. 17 and 18 illustrate that noise is perfectly canceled by the observer and the error synchronization is not affected. Moreover, despite the presence of noise, the unknown parameters  $a$  and  $b$  as well as the information signal, are well recovered by the observer (9) as is appreciated in Figs. 19, 20 and 21.



Fig. 20. Estimation of the parameter  $b$ .Fig. 21. Information signal  $m(t)$  and its estimate  $\hat{m}(t)$ .

## V. CONCLUDING REMARKS

We have presented a synchronization method based on robust adaptive estimation for partially linear systems affected by additive disturbances and driven by unknown inputs. The approach applies to a number of common chaotic circuit models used in transmission of encrypted information. We have established necessary and sufficient conditions for parameter estimation and information recovery. In particular, we presented two significant examples of chaotic systems used in a classic master-slave synchronization scheme for transmission of information. We have showed that the proposed methodology yields very good performance and robustness to measurement noise. In fact, neither stability nor the quality of information recovery are compromised by the perturbations.

## APPENDIX

*Proof of Fact 1:* The fact follows applying the mean value theorem for vectorial functions. After [25, Theorem A.3] for each  $w \in \mathbb{R}^s$ ,  $\sigma_1$  and  $\sigma_2 \in \Omega$  there exist  $\alpha \in (0, 1)$  and  $\xi := \sigma_2 + \alpha[\sigma_1 - \sigma_2]$  such that

$$\begin{aligned} w^\top [f_0(\sigma_1) - f_0(\sigma_2)] \\ = [\sigma_1 - \sigma_2]^\top \underbrace{\begin{bmatrix} \frac{\partial f_0}{\partial s_1} \Big|_{s=\xi} & \cdots & \frac{\partial f_0}{\partial s_n} \Big|_{s=\xi} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_0}{\partial s_1} \Big|_{s=\xi} & \cdots & \frac{\partial f_0}{\partial s_n} \Big|_{s=\xi} \end{bmatrix}}_{=: \Gamma(\xi)} w \end{aligned}$$

Since  $\partial f_0 / \partial s_i$  is continuous for all  $i$  and  $\xi \in \Omega$  it follows that every element in the matrix above is bounded from above. That is, there exists  $K_f$  such that  $|\Gamma(\xi)| \leq K_f$  for all  $\xi \in \Omega$ . The above holds for all  $\sigma_1, \sigma_2 \in \Omega$  hence for  $\sigma_1 := \sigma(a)$  and  $\sigma_2 := \sigma(b)$  for any  $a$  and  $b \in \mathbb{R}^n$ . Also, by definition of  $\sigma$  we have

$$|\sigma(a) - \sigma(b)| \leq |a - b| \quad \forall a, b \in \mathbb{R}^n.$$

Thus the claim. The proof for  $g$  follows with the obvious modifications.

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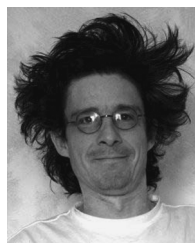
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